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Effect of rotation on plane waves in generalized thermo-elasticity with two relaxation times

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Abstract

The model of generalized thermo-elastic plane waves under the effect of rotation is studied using the theory of thermo-elasticity recently proposed by Green and Lindsay. The normal mode analysis is used to obtain the exact expressions for the temperature distribution, the displacement component and thermal stress. The resulting formulation is applied to two different concrete problems. The first deals with a thick plate subjected to a time-dependent heat source on each face. The second concerns the case of a heated punch moving across the surface of a semi-infinite thermo-elastic half-space subject to appropriate boundary conditions. Numerical results are given and illustrated graphically for each problem. Comparisons are made with the results predicted by the coupled theory and with the theory of generalized thermo-elasticity with two relaxation times in the absence of rotation.

Keywords: Rotation; Two relaxation times; Thermo-elasticity; Normal mode

1. Introduction

In recent years considerable interest has been shown in the study of plane thermo-elastic wave propagation in a non-rotating medium. The classical theory of thermo-elasticity is based on Fourier's law of heat conduction, which predicts an infinite speed of propagation of heat. This is physically absurd and many new theories have been proposed to eliminate this absurdity.

Two generalizations to the coupled theory were introduced. The first is due to Lord and Shulman (1967), who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. It also contains a new constant that acts as a relaxation time. Since the heat equation of this theory is of the wave type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motions and constitutive relations, remain the

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Nomenclature

λ, μ	Lame's constants
ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
T_o	reference temperature chosen so that $\left\lvert \frac{T - T_o}{T_o} \right\rvert \ll 1$
σ_{ij}	components of stress tensor
ε_{ij}	components of strain tensor
u_i	components of displacement vector
Ω	the rotation
k	thermal conductivity
c_o^2	$\frac{\lambda + 2\mu}{\rho}$
c_2	$\sqrt{\frac{\mu}{\rho}}$ velocity of transverse waves
β^2	$\frac{\lambda + 2\mu}{\mu}$
τ, v	two relaxation times
e	$\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right)$, the dilatation
α_t	coefficient of linear thermal expansion
γ	$(3\lambda + 2\mu)\alpha_t$
ε	$\gamma^2 T_o/\rho C_E(\lambda + 2\mu)$
η_o	$\rho C_E/k$

same as those for the coupled and the uncoupled theories. This theory was extended by Dhaliwal and Sherief (1980) to general anisotropic media in the presence of heat sources. Sherief and Dhaliwal (1981) solved a thermal shock problem. Both these problems are valid for short times. Recently, Sherief and Ezzat (1994) obtained the fundamental solution for this theory that is valid for all times.

The second generalization to the coupled theory of thermo-elasticity is what is known as the theory of thermo-elasticity with two relaxation times or the theory of temperature-rate-dependent thermo-elasticity. Müller (1971), in review of the thermo-dynamics of thermo-elastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations.

A generalization of this inequality was proposed by Green and Laws (1972). Green and Lindsay (1972) obtained an explicit version of the constitutive equations. These equations were also obtained independently by Şuhubi (1975). This theory contains two constants that act as relaxation times and modifies all the equations of the coupled theory, not only the heat equation. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Şuhubi (1986) studied wave propagation in finite cylinders. Ignaczak (1985) studied a strong discontinuity wave and obtained a decomposition theorem for this theory (Ignaczak, 1978). Dhaliwal and Rokne (1989) solved a thermal shock problem.

Using the Green–Lindsay theory, Agarwal (1979a,b) studied respectively thermo-elastic and magneto-thermo-elastic plane wave propagation in an infinite non-rotating medium. In a paper by Schoenberg and

Censor (1973), the propagation of plane harmonic waves in a rotating elastic medium without a thermal field has been studied. It was shown there that the rotation causes the elastic medium to be dispersive and an isotropic. Ezzat and Othman (2000) have established the model of the two-dimensional equations of generalized magneto-thermo-elasticity with two relaxation times in a perfect conducting medium without rotation.

It appears that little attention has been paid to the study of propagation of plane thermo-elastic waves in a rotating medium. Since most large bodies like the earth, the moon and other planets have an angular velocity it appears more realistic to study the propagation of plane thermo-elastic or magneto-thermo-elastic waves in a rotating medium with thermal relaxation. Using the Lord-Shulman theory, Roy Choudhuri and Debnath (1983a) studied the propagation of plane harmonic waves in an infinite conducting thermo-elastic solid permeated by a primary uniform magnetic field when the entire elastic medium is rotating with uniform angular velocity. The nature of the magneto-elastic waves in a rotating medium has been considered by Roy Choudhuri and Debnath (1983b).

In the present work we shall present the normal mode analysis to two-dimensional problems of generalized thermo-elasticity with two relaxation times under the effect of rotation in the context of the linearized theory of Green and Lindsay. The resulting formulation is applied to two concrete problems. The exact expressions for temperature, displacement and stress are obtained for the two problems considered.

2. Formulation of the problem

We consider an infinite isotropic, homogeneous, thermally conducting elastic medium. The medium is rotating uniformly with an angular velocity $\mathbf{\Omega} = \Omega\mathbf{n}$, where $\mathbf{n}$ is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame of reference has two additional terms (Schoenberg and Censor, 1973):

(i) Centripetal acceleration $\mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{u})$ due to the time-varying motion only;
(ii) The Coriolis acceleration $2\mathbf{\Omega} \wedge \dot{\mathbf{u}}$.

Here $\mathbf{u}$ is the dynamic displacement vector measured from a steady state deformed position and assumed to be small. These two terms do not appear in the equations for non-rotating media.

The fundamental equations of the generalized thermo-elasticity:

The constitutive law for the theory of generalized thermo-elasticity

$$\sigma_{ij} = \lambda e\delta_{ij} + 2\mu\varepsilon_{ij} - \gamma(T - T_o + v\dot{T})\delta_{ij}. \tag{1}$$

The heat conduction equation

$$kT_{,ii} = \rho C_E(\dot{T} + \tau\ddot{T}) + \gamma T_o\dot{e}. \tag{2}$$

The strain-displacement constitutive relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{and} \quad \varepsilon_{ii} = e = u_{i,i}. \tag{3}$$

The equations of motion, in the absence of body forces, are

$$\sigma_{ij,j} = \rho[\ddot{u}_i + \{\mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{u})\}_i + (2\mathbf{\Omega} \wedge \dot{\mathbf{u}})_i]. \tag{4}$$

Combining Eqs. (1), (3) and (4), we obtain the displacement equation of motion in the rotating frame of reference as

$$\rho[\ddot{\mathbf{u}} + \{\mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{u})\} + (2\mathbf{\Omega} \wedge \dot{\mathbf{u}})] = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u} - \gamma\nabla[T + v\dot{T}]. \tag{5}$$

From Eqs. (1) and (3) the stress components are given by

$$\sigma_{xx} = (\lambda + 2\mu)u_{,x} + \lambda v_{,y} - \gamma(T - T_o + \nu\dot{T}), \tag{6}$$

$$\sigma_{yy} = (\lambda + 2\mu)v_{,y} + \lambda u_{,x} - \gamma(T - T_o + \nu\dot{T}), \tag{7}$$

$$\sigma_{xy} = \mu(u_{,y} + v_{,x}), \tag{8}$$

$$\sigma_{zz} = \lambda e - \gamma(T - T_o + \nu\dot{T}). \tag{9}$$

From Eqs. (4) and (6)–(9), we get

$$\rho\left[\frac{\partial^2 u}{\partial t^2} - \Omega^2 u - 2\Omega\dot{v}\right] = (\lambda + \mu)\frac{\partial e}{\partial x} + \mu\nabla^2 u - \gamma\left(1 + \nu\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial x}, \tag{10}$$

$$\rho\left[\frac{\partial^2 v}{\partial t^2} - \Omega^2 v + 2\Omega\dot{u}\right] = (\lambda + \mu)\frac{\partial e}{\partial y} + \mu\nabla^2 v - \gamma\left(1 + \nu\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial y}. \tag{11}$$

For convenience, the following non-dimensional quantities are introduced:

$$x'_j = c_o\eta_o x_j, \quad u'_j = c_o\eta_o u_j, \quad t' = c_o^2\eta_o t, \quad \tau' = c_o^2\eta_o\tau, \quad \nu' = c_o^2\eta_o\nu, \quad \Omega' = \frac{\Omega}{c_o^2\eta_o},$$
$$\theta = \frac{\gamma(T - T_o)}{\lambda + 2\mu}, \quad \sigma'_{ij} = \frac{\sigma_{ij}}{\mu}. \tag{12}$$

In order to examine the effect of rotation and relaxation time on coupled elastic dilatational, shear and thermal waves, we get $\mathbf{\Omega} = (0, 0, \Omega)$, $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$, where Ω is a constant.

In terms of the non-dimensional quantities defined in Eq. (12), the above governing equations reduce to (dropping the dashes for convenience)

$$\beta^2\left[\frac{\partial^2 u}{\partial t^2} - \Omega^2 u - 2\Omega\dot{v}\right] = (\beta^2 - 1)\frac{\partial e}{\partial x} + \nabla^2 u - \beta^2\left(\frac{\partial\theta}{\partial x} + \nu\frac{\partial^2\theta}{\partial t\partial x}\right), \tag{13}$$

$$\beta^2\left[\frac{\partial^2 v}{\partial t^2} - \Omega^2 v + 2\Omega\dot{u}\right] = (\beta^2 - 1)\frac{\partial e}{\partial y} + \nabla^2 v - \beta^2\left(\frac{\partial\theta}{\partial y} + \nu\frac{\partial^2\theta}{\partial t\partial y}\right), \tag{14}$$

$$\nabla^2\theta = \left(\frac{\partial\theta}{\partial t} + \tau\frac{\partial^2\theta}{\partial t^2}\right) + \varepsilon\frac{\partial e}{\partial t} \tag{15}$$

and the components of the stress are

$$\sigma_{xx} = 2u_{,x} + (\beta^2 - 2)e - \beta^2(\theta + \nu\dot{\theta}), \tag{16}$$

$$\sigma_{yy} = (\beta^2 - 2)e + 2v_{,y} - \beta^2(\theta + \nu\dot{\theta}), \tag{17}$$

$$\sigma_{xy} = u_{,y} + v_{,x}, \tag{18}$$

$$\sigma_{zz} = (\beta^2 - 2)e - \beta^2(\theta + \nu\dot{\theta}). \tag{19}$$

In the subsequent analysis we are taking into consideration the case of low speed so that centrifugal stiffening effects can be neglected. By differentiating Eq. (13) with respect to x, and Eq. (14) with respect to y, then adding, we arrive at

$$\left[\nabla^2 - \frac{\partial^2}{\partial t^2} + \Omega^2\right] e = \left(1 + v\frac{\partial}{\partial t}\right)\nabla^2\theta + 2\Omega\frac{\partial \zeta}{\partial t}. \tag{20}$$

Differentiating Eq. (13) with respect to y, and Eq. (14) with respect to x, then subtracting, we arrive at

$$\left[\nabla^2 - \beta^2\left(\frac{\partial^2}{\partial t^2} - \Omega^2\right)\right]\zeta = -2\Omega\beta^2\frac{\partial e}{\partial t}, \tag{21}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplace's operator in a two-dimensional space and $\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$.

3. Normal mode analysis

The solution of the considered physical variable can be decomposed in terms of normal modes as the following form

$$[u, v, e, \zeta, \theta, \sigma_{ij}](x, y, t) = [u^*(y), v^*(y), e^*(y), \zeta^*(y), \theta^*(y), \sigma^*_{ij}(y)]\exp(\omega t + \mathrm{i}ax). \tag{22}$$

where ω is the (complex) time constant, $\mathrm{i} = \sqrt{-1}$ and a is the wave number in the x-direction and $u^*(y)$, $v^*(y)$, $e^*(y)$, $\zeta^*(y)$, $\theta^*(y)$ and $\sigma^*_{ij}(y)$ are the amplitude of the functions.

Using Eq. (22), we can obtain the following equations from Eqs. (15), (20) and (21) respectively

$$[D^2 - a^2 - \omega(1 + \tau\omega)]\theta^*(y) = \varepsilon\omega e^*(y), \tag{23}$$

$$[D^2 - a^2 - \omega^2 + \Omega^2]e^*(y) = (1 + v\omega)(D^2 - a^2)\theta^*(y) + 2\Omega\omega\zeta^*, \tag{24}$$

$$[D^2 - a^2 - \beta^2(\omega^2 - \Omega^2)]\zeta^*(y) = -2\beta^2\omega\Omega e^*, \tag{25}$$

where, $D = \frac{\partial}{\partial y}$.

Eliminating $\theta^*(y)$ and $\zeta^*(y)$ between Eqs. (23)–(25), we get the following sixth-order partial differential equation satisfied by $e^*(y)$

$$(D^6 - a_1D^4 + a_2D^2 - a_3)e^*(y) = 0, \tag{26}$$

where,

$$a_1 = 3a^2 + b_1, \tag{27}$$

$$a_2 = 3a^4 + 2a^2b_1 + b_2, \tag{28}$$

$$a_3 = a^6 + a^4b_1 + a^2b_2 + b_3, \tag{29}$$

$$b_1 = \varepsilon\omega_1 + \omega_2 + (\beta^2 + 1)\omega_3, \tag{30}$$

$$b_2 = \beta^2[\omega_3^2 + \omega_2\omega_3 + \varepsilon\omega_1\omega_3 + 4\omega^2\Omega^2] + \omega_2\omega_3, \tag{31}$$

$$b_3 = \beta^2\omega_2(\omega_3^2 + 4\omega^2\Omega^2), \tag{32}$$

$$\omega_1 = \omega(1 + v\omega), \quad \omega_2 = \omega(1 + \tau\omega), \quad \omega_3 = (\omega^2 - \Omega^2), \tag{33}$$

Eq. (26) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)e^*(y) = 0, \tag{34}$$

where, k_j, $j = 1, 2, 3$ are the roots of the following characteristic equation

$$k^6 - a_1 k^4 + a_2 k^2 - a_3 = 0. \tag{35}$$

The solution of Eq. (34) is given by:

$$e^*(y) = \sum_{j=1}^{3} e_j^*(y), \tag{36}$$

where $e_j^*(y)$ is the solution of the equation

$$(D^2 - k_j^2)e_j^*(y) = 0, \quad j = 1, 2, 3. \tag{37}$$

The solution of Eq. (37), which is bounded as $y \to \infty$, is given by

$$e_j^*(y) = G_j(a, \omega)e^{-k_j y}. \tag{38}$$

Substituting from Eq.(38) into Eq. (36), we obtain:

$$e^*(y) = \sum_{j=1}^{3} G_j(a, \omega)e^{-k_j y}. \tag{39}$$

In a similar manner, we get

$$\theta^*(y) = \sum_{j=1}^{3} G'_j(a, \omega)e^{-k_j y}, \tag{40}$$

$$\zeta^*(y) = \sum_{j=1}^{3} G''_j(a, \omega)e^{-k_j y}, \tag{41}$$

where $G_j(a, \omega)$, $G'_j(a, \omega)$ and $G''_j(a, \omega)$ are parameters depending on a, ω.
Substituting from Eqs. (39)–(41) into Eqs. (23) and (25), we obtain

$$G'_j(a, \omega) = \frac{\varepsilon\omega}{[k_j^2 - a^2 - \omega_2]} G_j(a, \omega), \quad j = 1, 2, 3, \tag{42}$$

$$G''_j(a, \omega) = \frac{-2\omega\Omega\beta^2}{[k_j^2 - a^2 - \beta^2\omega_3]} G_j(a, \omega), \quad j = 1, 2, 3. \tag{43}$$

Substituting from Eqs. (42) and (43) into Eqs. (40) and (41), respectively, we obtain

$$\theta^*(y) = \sum_{j=1}^{3} \frac{\varepsilon\omega}{[k_j^2 - a^2 - \omega_2]} G_j(a, \omega)e^{-k_j y}, \tag{44}$$

$$\zeta^*(y) = \sum_{j=1}^{3} \frac{-2\omega\Omega\beta^2}{[k_j^2 - a^2 - \beta^2\omega_3]} G_j(a, \omega)e^{-k_j y}. \tag{45}$$

Since,

$$\zeta^* = Du^* - \mathrm{i}av^*, \tag{46}$$

$$e^* = \mathrm{i}au^* + Dv^*. \tag{47}$$

In order to obtain the displacement u, in terms of Eq. (22), from Eqs. (13), (14), (46) and (47) we can obtain

$$u^*(y) = Be^{ay} + \sum_{j=1}^{3} \frac{1}{(k_j^2 - a^2)} \left[\mathrm{i}a + \frac{2\omega\Omega\beta^2 k_j}{[k_j^2 - a^2 - \beta^2\omega_3]} \right] G_j(a,\omega) e^{-k_j y}, \tag{48}$$

$$v^*(y) = -\mathrm{i}Be^{ay} - \sum_{j=1}^{3} \frac{1}{(k_j^2 - a^2)} \left[k_j - \frac{2\mathrm{i}a\omega\Omega\beta^2}{[k_j^2 - a^2 - \beta^2\omega_3]} \right] G_j(a,\omega) e^{-k_j y}, \tag{49}$$

where $B = 0$ to make Eqs. (48) and (49) are bounded as $y \to \infty$.

In terms of Eq. (22), substituting from Eqs. (39), (44), (45), (48) and (49) into Eqs. (16)–(19), respectively, we obtain the stress components in the form

$$\sigma_{xx}^*(y) = \sum_{j=1}^{3} \left\{ \frac{-2a^2}{(k_j^2 - a^2)} + 2\mathrm{i}a \left[\frac{2\omega\Omega\beta^2 k_j}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta^2\omega_3]} + \beta^2 - 2 - \frac{\varepsilon\omega_1\beta^2}{[k_j^2 - a^2 - \omega_2]} \right] \right\} G_j e^{-k_j y}, \tag{50}$$

$$\sigma_{yy}^*(y) = \sum_{j=1}^{3} \left\{ \frac{-4\mathrm{i}a\omega\Omega\beta^2 k_j}{[k_j^2 - a^2 - \beta^2\omega_3]} + \beta^2 - 2 + \frac{2k_j^2}{(k_j^2 - a^2)} - \frac{\varepsilon\omega_1\beta^2}{[k_j^2 - a^2 - \omega_2]} \right\} G_j e^{-k_j y}, \tag{51}$$

$$\sigma_{xy}^*(y) = -\sum_{j=1}^{3} \left\{ \frac{2\omega\Omega\beta^2 (k_j^2 + a^2)}{(k_j^2 - a^2)[k_j^2 - a^2 - \beta^2\omega_3]} + \frac{2\mathrm{i}ak_j}{(k_j^2 - a^2)} \right\} G_j e^{-k_j y}, \tag{52}$$

$$\sigma_{zz}^*(y) = \sum_{j=1}^{3} \left\{ \beta^2 - 2 - \frac{\varepsilon\omega_1\beta^2}{[k_j^2 - a^2 - \omega_2]} \right\} G_j e^{-k_j y}. \tag{53}$$

The normal mode analysis is, in fact, to look for the solution in Fourier transformed domain. This assumes that all the field quantities are sufficiently smooth on the real line such that the normal mode analysis of these functions exist.

4. Applications

Problem I. A plate subjected to time-dependent heat sources on both sides (Sherief and Anwar, 1986).

We shall consider a homogeneous isotropic thermo-elastic infinite conductive thick flat plate of a finite thickness $2L$ occupying the region G given by:

$$G = \{(x,y,z) | -\infty < x < \infty, -L \leqslant y \leqslant L, -\infty < z < \infty\},$$

with the middle surface of the plate coinciding with the plane $y = 0$.

The boundary conditions of the problem are taken as:

(i) The thermal boundary condition

$$q_n + h_o\theta = r(x,t) \quad \text{on } y = \pm L, \tag{54}$$

where q_n denotes the normal component of the heat flux vector, h_o is Biot's number and $r(x,t)$ represents the intensity of the applied heat sources.

(ii) The normal and tangential stress components are zero on both surfaces of the plate; thus,

$$\sigma_{yy} = 0 \quad \text{on } y = \pm L, \tag{55}$$

$$\sigma_{xy} = 0 \quad \text{on } y = \pm L. \tag{56}$$

Eqs. (55) and (56) in the normal mode form together with Eqs. (51) and (52) respectively give:

$$S_1G_1\cosh(k_1L)+S_2G_2\cosh(k_2L)+S_3G_3\cosh(k_3L)=0, \tag{57}$$

$$N_1G_1\sinh(k_1L)+N_2G_2\sinh(k_2L)+N_3G_3\sinh(k_3L)=0. \tag{58}$$

We now make use of the generalized Fourier's law of heat conduction in the non-dimensional form, namely,

$$q_n+\tau\frac{\partial q_n}{\partial t}=-\frac{\partial\theta}{\partial n}. \tag{59}$$

In terms of Eq. (22), from Eq. (59), we obtain

$$q_n^*=-\frac{1}{(1+\tau\omega)}\frac{\partial\theta^*}{\partial n}. \tag{60}$$

Combining Eqs. (44), (59) and (60) we arrive at

$$A_1G_1\cosh(k_1L)+A_2G_2\cosh(k_2L)+A_3G_3\cosh(k_3L)=r^*(1+\tau\omega), \tag{61}$$

where,

$$A_j=\frac{\varepsilon\omega}{\alpha_j}[-k_j\sinh(k_jL)+h_o(1+\tau\omega)\cosh(k_jL)],\quad j=1,2,3, \tag{62}$$

$$\alpha_j=[k_j^2-a^2-\omega_2],\quad \beta_j=[k_j^2-a^2-\beta^2\omega_3],\quad j=1,2,3, \tag{63}$$

$$S_j=(\alpha_{j1}-\mathrm{i}\beta_{j1}),\quad j=1,2,3, \tag{64}$$

$$N_j=(\alpha_{j2}+\mathrm{i}\beta_{j2}),\quad j=1,2,3, \tag{65}$$

$$\alpha_{j1}=\left[\beta^2-2+\frac{2k_j^2}{(k_j^2-a^2)}-\frac{\varepsilon\omega_1\beta^2}{\alpha_j}\right],\quad j=1,2,3, \tag{66}$$

$$\beta_{j1}=\frac{4a\omega\Omega\beta^2k_j}{\beta_j},\quad j=1,2,3, \tag{67}$$

$$\alpha_{j2}=\frac{2\omega\Omega\beta^2(k_j^2+a^2)}{(k_j^2-a^2)\beta_j},\quad j=1,2,3, \tag{68}$$

$$\beta_{j2}=\frac{2ak_j}{(k_j^2-a^2)},\quad j=1,2,3. \tag{69}$$

Eqs. (57), (58) and (61) can be solved for the three unknowns G_1, G_2 and G_3.

$$G_1=\frac{(1+\tau\omega)r^*}{\varepsilon\omega(\Delta_1^2+\Delta_2^2)\cosh(k_1L)}[(\lambda_1\Delta_1+\lambda_2\Delta_2)+\mathrm{i}(\lambda_2\Delta_1-\lambda_1\Delta_2)], \tag{70}$$

$$G_2=\frac{-(1+\tau\omega)r^*}{(\Delta_1^2+\Delta_2^2)\cosh(k_1L)}[(\lambda_3\Delta_1+\lambda_4\Delta_2)+\mathrm{i}(\lambda_4\Delta_1-\lambda_3\Delta_2)], \tag{71}$$

$$G_3=\frac{(1+\tau\omega)r^*}{(\Delta_1^2+\Delta_2^2)\cosh(k_1L)}[(\lambda_5\Delta_1+\lambda_6\Delta_2)+\mathrm{i}(\lambda_6\Delta_1-\lambda_5\Delta_2)], \tag{72}$$

$$\lambda_1 = (\alpha_{21}\alpha_{32} + \beta_{21}\beta_{32})\tanh(k_3L) - (\alpha_{31}\alpha_{22} + \beta_{31}\beta_{22})\tanh(k_2L), \tag{73}$$

$$\lambda_2 = (\alpha_{21}\beta_{32} - \alpha_{32}\beta_{21})\tanh(k_3L) - (\alpha_{31}\beta_{22} - \alpha_{22}\beta_{31})\tanh(k_2L), \tag{74}$$

$$\lambda_3 = \left[\frac{k_3\alpha_{22}}{\alpha_3} - \frac{k_2\alpha_{32}}{\alpha_2}\right]\tanh(k_2L)\tanh(k_3L) + h_o(1+\tau\omega)\left[\frac{\alpha_{32}}{\alpha_2}\tanh(k_3L) - \frac{\alpha_{22}}{\alpha_3}\tanh(k_2L)\right], \tag{75}$$

$$\lambda_4 = \left[\frac{k_3\beta_{22}}{\alpha_3} - \frac{k_2\beta_{32}}{\alpha_2}\right]\tanh(k_2L)\tanh(k_3L) + h_o(1+\tau\omega)\left[\frac{\beta_{32}}{\alpha_2}\tanh(k_3L) - \frac{\beta_{22}}{\alpha_3}\tanh(k_2L)\right], \tag{76}$$

$$\lambda_5 = \frac{k_3\alpha_{21}}{\alpha_3}\tanh(k_3L) - \frac{k_2\alpha_{31}}{\alpha_2}\tanh(k_2L) + h_o(1+\tau\omega)\left(\frac{\alpha_{31}}{\alpha_2} - \frac{\alpha_{21}}{\alpha_3}\right), \tag{77}$$

$$\lambda_6 = \frac{k_2\beta_{31}}{\alpha_2}\tanh(k_2L) - \frac{k_3\beta_{21}}{\alpha_3}\tanh(k_3L) + h_o(1+\tau\omega)\left(\frac{\beta_{21}}{\alpha_1} - \frac{\beta_{31}}{\alpha_2}\right), \tag{78}$$

$$\Delta_1 = \frac{\lambda_1}{\alpha_1}[-k_1\tanh(k_1L) + h_o(1+\tau\omega)] - \alpha_{11}\lambda_3 + \alpha_{12}\lambda_5\tanh(k_1L) - \beta_{11}\lambda_4 - \beta_{12}\lambda_6\tanh(k_1L), \tag{79}$$

$$\Delta_2 = \frac{\lambda_2}{\alpha_1}[-k_1\tanh(k_1L) + h_o(1+\tau\omega)] - \alpha_{11}\lambda_4 + \alpha_{12}\lambda_6\tanh(k_1L) + \beta_{11}\lambda_3 + \beta_{12}\lambda_5\tanh(k_1L). \tag{80}$$

Problem II. A time-dependent heat punch across the surface of semi-infinite thermo-elastic half space (Nowacki, 1975).

We consider a homogeneous isotropic thermo-elastic solid occupying the region G^* given by $G^* = \{(x,y,z)| -\infty < x < \infty, 0 \leqslant y, -\infty < z < \infty\}$.

In the physical problem, we should suppress the positive exponentials that are unbounded at infinity.

The constants G_1^*, G_2^* and G_3^* have to be chosen such that the boundary conditions on the surface $y = 0$ take the form

$$\theta(x,y,t) = n(x,t) \quad \text{on } y = 0, \tag{81}$$

$$\sigma_{yy}(x,y,t) = P(x,t) \quad \text{on } y = 0, \tag{82}$$

$$\sigma_{xy}(x,y,t) = 0 \quad \text{on } y = 0, \tag{83}$$

where n, P are given functions of x and t.

Eqs. (81)–(83) in the normal mode form together with Eqs. (44), (51) and (52) respectively, give

$$L_1G_1^* + L_2G_2^* + L_3G_3^* = n^*(a,\omega), \tag{84}$$

$$S_1G_1^* + S_2G_2^* + S_3G_3^* = P^*(a,\omega), \tag{85}$$

$$N_1G_1^* + N_2G_2^* + N_3G_3^* = 0. \tag{86}$$

Eqs. (84)–(86) can be solved for the three unknowns G_1^*, G_2^* and G_2^* one obtains

$$G_1^* = \frac{1}{(\Delta_3^2 + \Delta_4^2)}[(\lambda_7\Delta_3 + \lambda_8\Delta_4) + \mathrm{i}(\lambda_8\Delta_3 - \lambda_7\Delta_4)], \tag{87}$$

$$G_2^* = \frac{1}{(\Delta_3^2 + \Delta_4^2)}[(\lambda_9\Delta_3 + \lambda_{10}\Delta_4) + \mathrm{i}(\lambda_{10}\Delta_3 - \lambda_9\Delta_4)], \tag{88}$$

$$G_3^* = \frac{1}{(\Delta_3^2 + \Delta_4^2)}[(\lambda_{11}\Delta_3 + \lambda_{12}\Delta_4) + \mathrm{i}(\lambda_{12}\Delta_3 - \lambda_{11}\Delta_4)], \tag{89}$$

$$L_j = \frac{\varepsilon\omega}{\alpha_j}, \quad j = 1, 2, 3, \tag{90}$$

$$\lambda_7 = n^*(\alpha_{21}\alpha_{32} + \beta_{21}\beta_{32} - \alpha_{31}\alpha_{22} - \beta_{31}\beta_{22}) - P^*(L_2\alpha_{32} - L_3\alpha_{22}), \tag{91}$$

$$\lambda_8 = n^*(\alpha_{21}\beta_{32} - \alpha_{32}\beta_{21} + \alpha_{22}\beta_{31} - \alpha_{31}\beta_{22}) - P^*(L_2\beta_{32} - L_3\beta_{22}), \tag{92}$$

$$\lambda_9 = P^*(L_1\alpha_{32} - L_3\alpha_{12}) - n^*(\alpha_{11}\alpha_{32} + \beta_{11}\beta_{32} - \alpha_{31}\alpha_{12} - \beta_{31}\beta_{12}), \tag{93}$$

$$\lambda_{10} = P^*(L_1\beta_{32} - L_2\beta_{12}) - n^*(\alpha_{11}\beta_{32} - \alpha_{32}\beta_{11} + \alpha_{12}\beta_{31} - \alpha_{31}\beta_{12}), \tag{94}$$

$$\lambda_{11} = n^*(\alpha_{11}\alpha_{22} + \beta_{11}\beta_{22} - \alpha_{21}\alpha_{12} - \beta_{21}\beta_{12}) - P^*(L_1\alpha_{22} - L_2\alpha_{12}), \tag{95}$$

$$\lambda_{12} = n^*(\alpha_{11}\beta_{22} - \alpha_{22}\beta_{11} - \alpha_{21}\beta_{12} + \alpha_{12}\beta_{21}) - P^*(L_1\beta_{22} - L_2\beta_{12}), \tag{96}$$

$$\begin{aligned}\Delta_3 = {} & L_1(\alpha_{21}\alpha_{32} + \beta_{21}\beta_{32} - \alpha_{31}\alpha_{22} - \beta_{31}\beta_{22}) - L_2(\alpha_{11}\alpha_{32} - \beta_{11}\beta_{32} - \alpha_{31}\alpha_{12} - \beta_{31}\beta_{12}) \\ & + L_3(\alpha_{11}\alpha_{22} + \beta_{11}\beta_{22} - \alpha_{21}\alpha_{12} - \beta_{21}\beta_{12}),\end{aligned} \tag{97}$$

$$\begin{aligned}\Delta_4 = {} & L_1(\alpha_{21}\beta_{32} - \alpha_{32}\beta_{21} + \alpha_{22}\beta_{31} - \alpha_{31}\beta_{22}) - L_2(\alpha_{11}\beta_{32} - \alpha_{32}\beta_{11} + \alpha_{12}\beta_{31} - \alpha_{31}\beta_{12}) \\ & + L_3(\alpha_{11}\beta_{22} - \alpha_{22}\beta_{11} - \alpha_{21}\beta_{12} + \alpha_{12}\beta_{21}).\end{aligned} \tag{98}$$

5. Numerical results

The copper material was chosen for the purpose of numerical evaluations. Since we have $\omega = \omega_o + \mathrm{i}\zeta$, where 'i' is imaginary unit, $e^{\omega t} = e^{\omega_o t}(\cos\zeta t + \mathrm{i}\sin\zeta t)$ and for small values of time, we can take $\omega = \omega_o$ (real). The numerical constants of the problems were taken as: $\varepsilon = 0.0168$, $\beta^2 = 3.5$, $\rho = 8954$, $\tau = 0.02$, $v = 0.03$, $\omega_o = 1$, $a = 1.2$, $h_o = 50$, $r^* = 1$, $P^* = 100$, $n^* = 50$. The computations were carried out for two values of time $t = 0.1$ and $t = 0.3$. The numerical techniques, outlined above, and used for the real part of $\theta(x, y, t)$ and $u(x, y, t)$ were calculated on the surface $y = 2$ and on the middle plane $y = 0$ for problem I, where $L = 4$, while for problem II on $y = 6$ for two different values of $\Omega = 0$ and $\Omega = 0.01$. The results are shown in Figs. 1–12. The graph shows the four curves predicted by different theories of thermo-elasticity. In these figures, the dashed lines represent the solution corresponding to using the Coupled Theory (CD) of heat conduction ($\tau = 0, v = 0$), the solid lines represent the solution for *G–L* theory ($\tau = 0.02, v = 0.03$). It can be seen from these figures that the rotation acts to decrease the magnitude of the real part of the temperature and increase the magnitude of the real part of the displacement. We notice also, that results for the temperature when the relaxation time appears in the heat equation are distinctly different from those the relaxation time is not mentioned in the equation of motion and heat equation. This is due to the fact that thermal waves in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non-Fourier case. This demonstrates clearly the difference between the coupled and the generalized

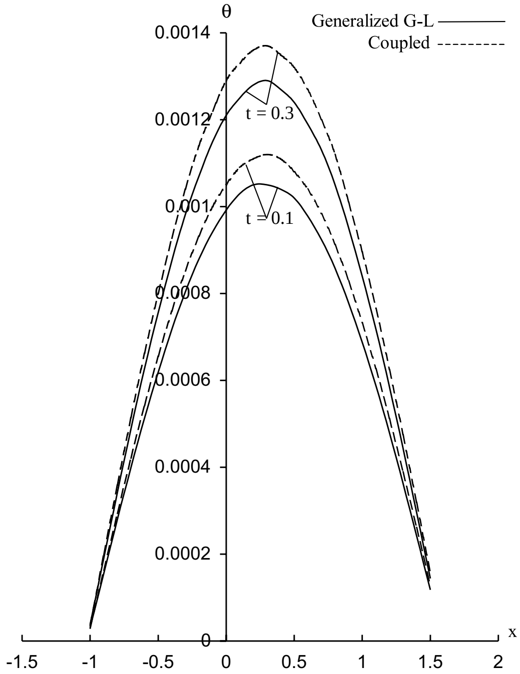

Fig. 1. Temperature distribution θ on the surface of problem I at $\Omega = 0.01$.

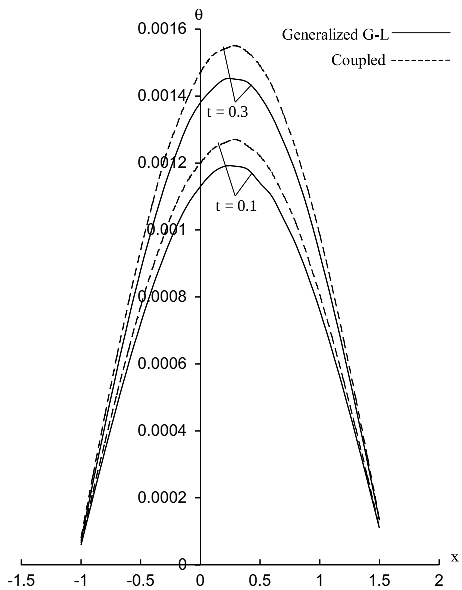

Fig. 2. Temperature distribution θ on the surface of problem I at $\Omega = 0$.

theories of thermo-elasticity. It is clear from Figs. 1–12 the effect of the rotation on the field quantities in the two specific problems.

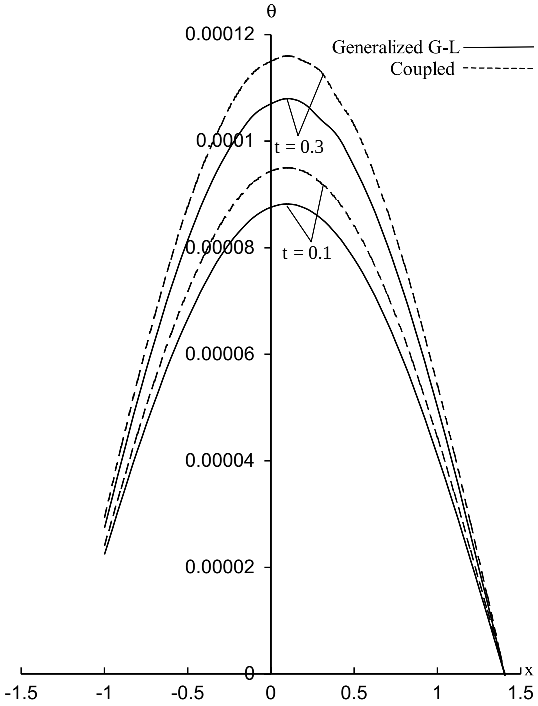

Fig. 3. Temperature distribution θ on the middle plane of problem I at $\Omega = 0.01$.

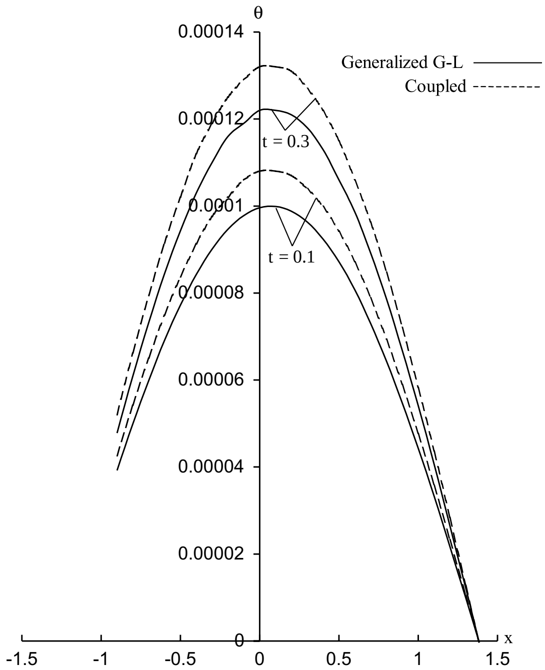

Fig. 4. Temperature distribution θ on the middle plane of problem I at $\Omega = 0$.

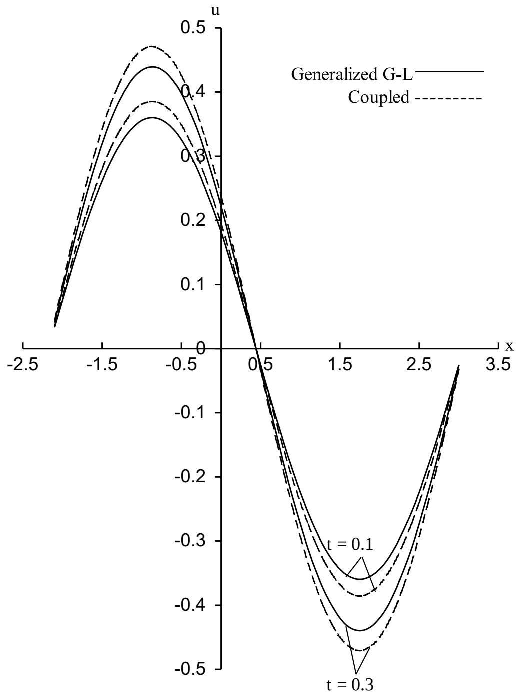

Fig. 5. Displacement distribution u on the surface of problem I at $\Omega = 0.01$.

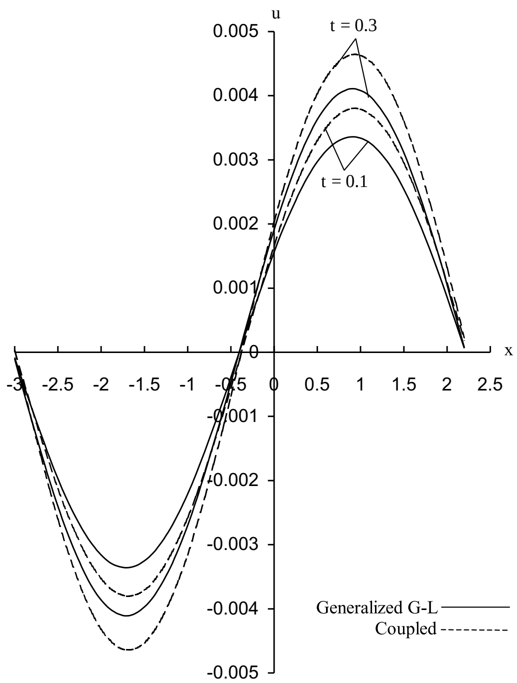

Fig. 6. Displacement distribution u on the surface of problem I at $\Omega = 0$.

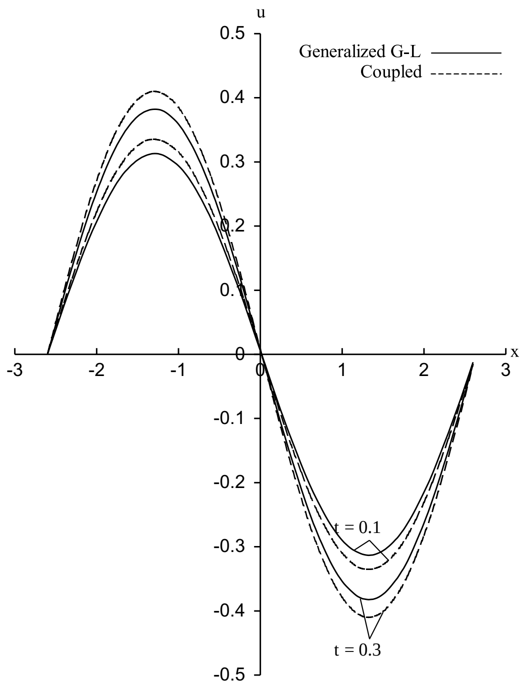

Fig. 7. Displacement distribution u on the middle plane of problem I at $\Omega = 0.01$.

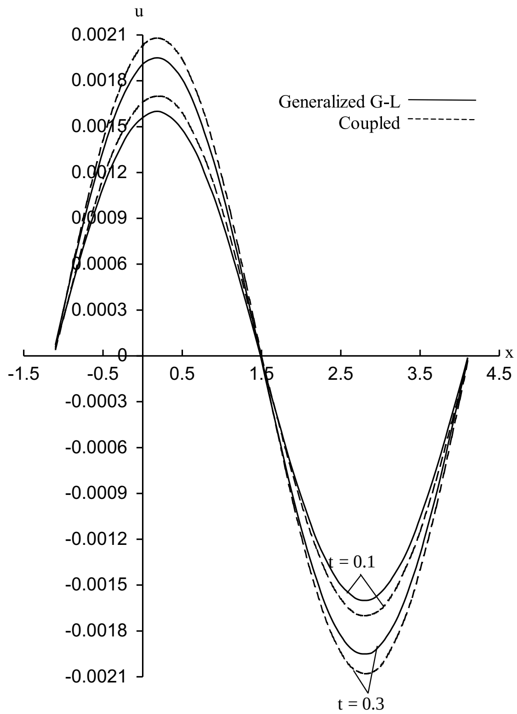

Fig. 8. Displacement distribution u on the middle plane of problem I at $\Omega = 0$.

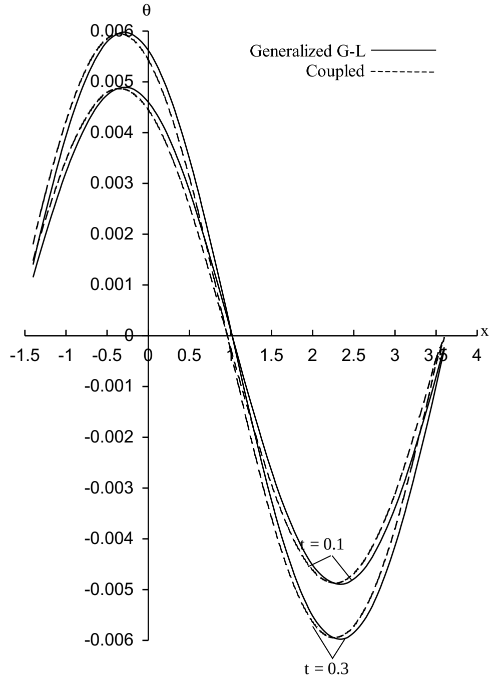

Fig. 9. Temperature distribution θ for $y = 6$ of problem II at $\Omega = 0.01$.

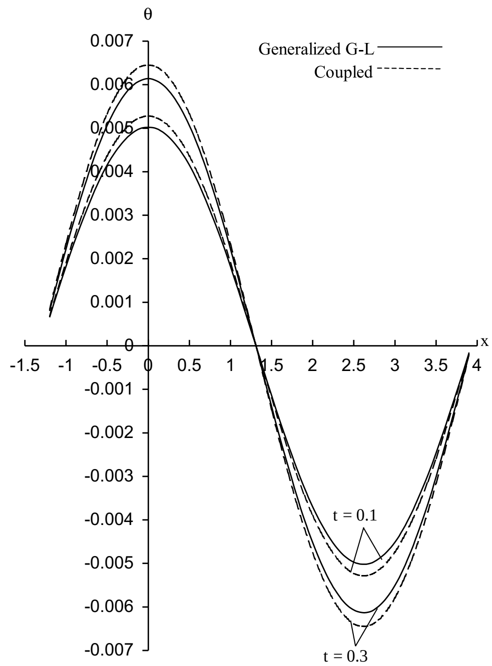

Fig. 10. Temperature distribution θ for $y = 6$ of problem II at $\Omega = 0$.

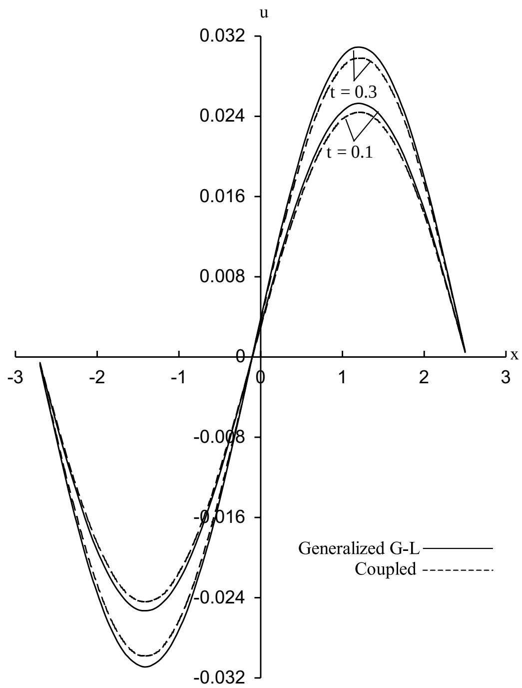

Fig. 11. Displacement distribution u for $y = 6$ of problem II at $\Omega = 0.01$.

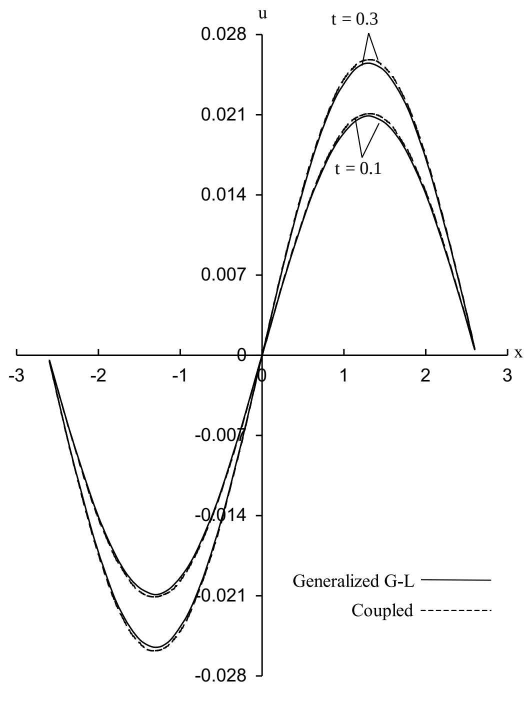

Fig. 12. Displacement distribution u for $y = 6$ of problem II at $\Omega = 0$.

6. Concluding remarks

Due to the complicated nature of the governing equations for generalized thermo-elasticity, with two relaxation times, few attempts have been made to solve problems in this field read (Nowacki, 1975). These attempts utilized an approximate method that is valid only for a specific range of some parameters.

In this work the method of normal mode analysis is introduced in the field of thermo-elasticity and applied to two specific problems in which the temperature, displacement and stress are coupled. This method gives exact solutions without any assumed restrictions on temperature, displacement and stress distributions.

The normal mode analysis is applied to a wide range of problems in different branches as that shown in (Ezzat and Othman, 2000; Othman, 2002). It can be applied to boundary-layer problems, which are described by the linearized Navier–Stokes equations in hydrodynamics as that shown in (Othman, 2001; Othman and Ezzat, 2001; Othman and Sweilam, 2002).

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